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Semantic Alternatives in Brouwer–Zadeh Logics

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In this paper we investigate an algebraic semantics for a particular kind of Brouwer–Zadeh logic (BZL*). We prove that this semantics and the orthopair semantics (based on certainly-yes and certainly-no domains) characterize BZL*. Further, we prove that every BZ* lattice can be embedded into a complete BZ* lattice.

INTRODUCTION

Brouwer–Zadeh logic (BZL) is a kind of quantum logic first investigated in Cattaneo and Nisticò (1989) and in Giuntini (1990, 1991). In contrast to standard quantum logic (*orthologic*, *orthomodular quantum logic*), BZL has two kinds of negation: a fuzzy-like negation and an intuitionistic-like negation.

As is well known, standard quantum logic, created in the thirties by Birkhoff and von Neumann, can be considered as a faithful abstraction of the structure of all closed subspaces (equivalently, projectors) of a Hilbert space. This structure is a complete orthomodular nondistributive lattice. The projectors of a Hilbert space are interpreted, after Birkhoff and von Neumann, as the properties pertaining to a quantum physical system.

Recently, some objections against the identification of the properties of a quantum physical system with projectors have been put forward. In particular, in the so-called operational approach to quantum mechanics, projectors are replaced by *effects* as the mathematical interpretation of the unsharp properties of a physical system. Effects are bounded self-adjoint operators between the null and the identity operators. As proved by Cattaneo and Nisticò (1989), the class of all effects of a Hilbert space determines a Brouwer–Zadeh poset, i.e., a bounded poset with two kinds of negations

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linked by a certain condition. Giuntini (1990) proved that any Brouwer–Zadeh poset can be embedded into a complete Brouwer–Zadeh lattice so that one can construct a logic based on Brouwer–Zadeh lattices. This logic can be considered as a relevant logical abstraction of an interesting algebraic structure arising from the operational approach to quantum mechanics (Giuntini, 1990). Giuntini (1991) proved that BZL can be equivalently characterized by means of an algebraic and a Kripkean semantics.

Recently, a new semantics for Brouwer–Zadeh logic has been proposed by Cattaneo and Nisticò (1989). Cattaneo *et al.* (1990) proved that such a semantics characterizes a logic which is stronger than BZL. We will call this logic BZL*. Cattaneo *et al.* (1990) proved a completeness theorem for BZL* with respect to this semantics. In this paper, we will show that BZL* can be equivalently characterized also by means of an algebraic semantics.

1. BROUWER–ZADEH LATTICES

A Brouwer–Zadeh lattice (BZ lattice) is a bounded lattice with two orthocomplementations linked by an interconnection rule. The first complement represents a generalization of the usual orthocomplementation of fuzzy set theory, while the second one is a generalization of the intuitionistic complement.

Definition 1.1. An *involutive bounded poset* (*lattice*) is a structure $\mathfrak{P} = \langle P, \leq, ^{\perp}, \mathbf{0}, \mathbf{1}\rangle$ satisfying the following conditions:

(i) $\langle P, \leq, \mathbf{0}, \mathbf{1}\rangle$ is a partially ordered set (poset) (lattice) with *maximum* ($\mathbf{1}$) and *minimum* ($\mathbf{0}$).

(ii) $^{\perp}$ is a 1-ary operation on P (the fuzzy-like complement) which satisfies the following conditions $\forall a, b \in P$:

(a) $a^{\perp\perp} = a$.
(b) If $a \leq b$, then $b^{\perp} \leq a^{\perp}$.

Two elements a, b of an involutive bounded poset are said to be *orthogonal* ($a \perp b$) iff $a \leq b^{\perp}$. The *sup* and the *inf* of two elements a, b, when they exist, will be denoted by $a \sqcup b$ and $a \sqcap b$, respectively.

Definition 1.2. A *regular involutive bounded poset* (*lattice*) is an involutive bounded poset (lattice) $\mathfrak{P} = \langle P, \leq, ^{\perp}, \mathbf{1}, \mathbf{0}\rangle$ satisfying the following condition:

$$\forall a, b \in P: \quad \text{if } a \perp a \text{ and } b \perp b, \quad \text{then } a \perp b$$

(*regularity condition*)

Example 1.1. Let $E(\mathfrak{H})$ be the set of all *effects* of a complex Hilbert space whose inner product is $(.,.)$. An effect is a bounded linear positive operator E on $\mathfrak{H}$ such that $\forall\varphi\in\mathfrak{H}$: $(E\varphi, \varphi)\leq\|\varphi\|^2$. Then the structure $\mathfrak{E}(\mathfrak{H}) = \langle E(\mathfrak{H}), \leq, {}^\perp, \mathbf{0}, \mathbf{1}\rangle$, where:

(i) $\forall E, F\in E(\mathfrak{H})$: $E\leq F$ iff $\forall\varphi\in\mathfrak{H}$: $(E\varphi, \varphi)\leq(F\varphi, \varphi)$.

(ii) $\mathbf{1}$ and $\mathbf{0}$ are the identity ($\mathbb{1}$) and the null ($\mathbb{0}$) operators, respectively.

(iii) $\forall F\in E(\mathfrak{H})$: $F^\perp := \mathbf{1} - F$,

is a regular involutive bounded poset which is not a lattice.

Example 1.2. Let $L := [0, 1]\subset\mathbb{R}$. Then the structure

$$\mathfrak{L}_{[0,1]} = \langle L, \leq, {}^\perp, \mathbf{1}, \mathbf{0}\rangle$$

where $\leq$ is the natural order of $\mathbb{R}$, $\mathbf{1}$ is 1, $\mathbf{0}$ is 0, and $\forall a\in L$, $a^\perp := \mathbf{1} - a$, is an involutive bounded lattice, where

$$a\sqcap b = \mathrm{Min}(\{a, b\})$$

and

$$\mathbf{a}\sqcup\mathbf{b} = \mathrm{Max}(\{a, b\})$$

Definition 1.3. An involutive bounded lattice $\mathfrak{L} = \langle L, \leq, {}^\perp, \mathbf{1}, \mathbf{0}\rangle$ satisfies the *Kleene property* iff for any $a, b\in P$, the following condition is satisfied:

$$a\sqcap a^\perp\leq b\sqcup b^\perp$$

Lemma 1.1. An involutive bounded lattice is regular iff it satisfies the Kleene property.

Definition 1.4. An *orthoposet* (*ortholattice*) is an involutive bounded poset $\mathfrak{P} = \langle P, \leq, {}^\perp, \mathbf{1}, \mathbf{0}\rangle$ satisfying the following condition for any $a\in L$:

$$a\sqcap a^\perp = \mathbf{0}$$

Lemma 1.2. Let $\mathfrak{P} = \langle P, \leq, {}^\perp, \mathbf{1}, \mathbf{0}\rangle$ be an involutive bounded poset. Then the following conditions are equivalent:

(i) $\{a\in P/a\perp a\} = \{\mathbf{0}\}$.

(ii) $\mathfrak{P}$ is an orthoposet.

Definition 1.5. An *orthomodular lattice* is an ortholattice

$$\mathfrak{L} = \langle L, \leq, {}^\perp, \mathbf{1}, \mathbf{0}\rangle$$

satisfying the following condition for any $a, b\in L$:

$$a\sqcap(a^\perp\sqcup(a\sqcap b))\leq b$$

Definition 1.6. A *Brouwer–Zadeh poset* (*BZ poset*) (*BZ lattice*) is a structure $\mathfrak{P}=\langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ which satisfies the following conditions:

(i) $\langle P, \leq, {}^{\perp}, \mathbf{1}, \mathbf{0}\rangle$ is a regular involutive bounded poset (lattice).

(ii) ${}^{\sim}$ is a 1-ary operation on P (the intuitionistic-like complement) which satisfies the following conditions:

(a) $a \leq a^{\sim\sim}$.
(b) If $a \leq b$, then $b^{\sim} \leq a^{\sim}$.
(c) $a \sqcap a^{\sim} = \mathbf{0}$.

(iii) $\forall a \in P$: $a^{\sim\perp} = a^{\sim\sim}$.
(iv) $\mathbf{1} = \mathbf{0}^{\sim} = \mathbf{0}^{\perp}$.

Lemma 1.3. Let $\mathfrak{P}=\langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ be a BZ poset. The following properties hold true:

(i) $a^{\sim} \leq a^{\perp}$.
(ii) If $a \sqcup b$ exists in P, then $a^{\sim} \sqcap b^{\sim}$ exists in P and $(a \sqcup b)^{\sim} = a^{\sim} \sqcap b^{\sim}$.

Lemma 1.4. Let $\mathfrak{P}=\langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ be a BZ poset. Then the following conditions are equivalent:

(i) $a = a^{\sim\sim}$.
(ii) $a^{\sim} = a^{\perp}$.
(iii) $a = a^{\sim\perp}$.
(iv) $a = a^{\perp\sim}$.

Lemma 1.5. Let $\mathfrak{P}=\langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ be a BZ poset. Then the set $P_e := \{a \in P / a = a^{\sim\sim}\}$ is nonempty since $\mathbf{0}, \mathbf{1} \in P_e$ and moreover:

(i) $a^{\sim} = a^{\perp}$, $\forall a \in P_e$. The set P_e endowed with the restriction of the partial order $\leq$ defined on P is an orthoposet bounded by $\mathbf{0}, \mathbf{1}$, with respect to the orthocomplementation ${}^{\sim}$: $P_e \to P_e$.

(ii) If $\mathfrak{P}$ is a lattice, then P_e is closed with respect to inf and sup and these are just the inf and sup in P.

The elements of P_e are called *exact elements* of $\mathfrak{P}$ and the elements of P/P_e *are called fuzzy elements* of $\mathfrak{P}$.

Definition 1.7. Let $\mathfrak{P}=\langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ be a BZ poset (lattice). An element $a \in P$ is said to be a *half element* of $\mathfrak{P}$ iff it satisfies the following conditions:

(i) $a \leq a^{\perp}$.
(ii) $\forall b \in P$: if $b \perp b$, then $b \leq a$

It is easy to see that if a BZ poset has a half element, then this is unique. Such an element will be denoted by $\mathbf{1/2}$.

Example 1.3. The structure $\mathfrak{E}(\mathfrak{H}) = \langle E(\mathfrak{H}), \leq, {}^{\perp}, \mathbf{0}, \mathbf{1}, \mathbf{1/2}\rangle$, where:
(i) $\langle E(\mathfrak{H}), \leq, {}^{\perp}, \mathbf{0}, \mathbf{1}\rangle$ is the involutive bounded poset of Example 1.1.
(ii) $\mathbf{1/2} := 1/2 \cdot \mathbf{1} = 1/2 \cdot \mathbb{1}$.
(iii) $\forall F \in E(\mathfrak{H})$: $F^{\sim} := P_{\mathrm{Ker}(F)}$, where $\mathrm{Ker}(F) := \{\varphi \in \mathfrak{H} / F\varphi = \underline{0}\}$ ($\underline{0}$ is the origin vector) and $P_{\mathrm{Ker}(F)}$ is the projector of $\mathfrak{H}$ associated with the closed subspace of $\mathfrak{H}$ determined by $\mathrm{Ker}(F)$, is a BZ poset with the half element.

It should be noted that the set of all exact elements of $\mathfrak{E}(\mathfrak{H})$ coincides with the set of all projectors of $\mathfrak{H}$. This set, as is well known, determines a complete orthomodular lattice. This example shows that even if a BZ poset is not a lattice, the orthoposet of all exact elements can be a complete lattice.

Example 1.4. The structure $\mathfrak{L}_{[0,1]} = \langle L, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}, \mathbf{1/2}\rangle$, where (i) $\langle L, \leq, {}^{\perp}, \mathbf{1}, \mathbf{0}\rangle$ is the involutive bounded lattice of Example 1.2, (ii) $\mathbf{1/2} := 1/2 \in \mathbb{R}$, and (iii) $\forall a \in L$:

$$a^{\sim} := \begin{cases} \mathbf{1} & \text{if } \ a = \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

is a BZ lattice with the half element.

Example 1.5. Let $P(\mathfrak{H})$ be the class of all projectors of a Hilbert space and let $P_{1/2}(\mathfrak{H}) := P(\mathfrak{H}) \cup \{1/2 \cdot \mathbb{1}\}$. Then the structure

$$\mathfrak{P}_{1/2}(\mathfrak{H}) = \langle P_{1/2}(\mathfrak{H}), \leq, {}^{\perp}, {}^{\sim}, \mathbf{0}, \mathbf{1}\rangle$$

where $\leq$ is the partial order of $\mathfrak{E}(\mathfrak{H})$ (see Example 1.3) restricted to $P_{1/2}(\mathfrak{H})$, and ${}^{\perp}$ and ${}^{\sim}$ are the operations of $\mathfrak{E}(\mathfrak{H})$ restricted to $P_{1/2}(\mathfrak{H})$, is a BZ lattice with the half element; further, $1/2 \cdot \mathbb{1} \sqcap F = \mathbf{0}$ and $1/2 \cdot \mathbb{1} \sqcup F = \mathbf{1}$, for any nontrivial projector F.

2. ALGEBRAIC SEMANTICS FOR BZL

In this section, we will present an algebraic semantics for Brouwer–Zadeh logic (BZL) and a calculus for BZL. For the proofs of the results, see Giuntini (1991).

The language of BZL contains a countable set $p_1, \ldots, p_n, \ldots$ of sentential letters, two sentential letters λ (the *falsity*) and χ (the *indeterminacy*), and three primitive connectives $\wedge$ (and), $\neg$ ("fuzzy" not), and $\sim$ ("intuitionistic" not). We will use $\alpha, \beta, \ldots$ as metavariables ranging over formulas of the language of BZL. Disjunction ($\vee$) is defined by the De Morgan law: $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$. A *necessity* operator (L) is defned as $L\alpha := \sim\neg\alpha$. A *possibility operator* (M) is defined as $M\alpha := \sim\sim\alpha$.

Definition 2.1. An *algebraic realization for* BZL is a pair $\mathfrak{U} = \langle \mathfrak{L}, v\rangle$, where $\mathfrak{L}$ is a BZ lattice with the half element and v is a valuation-function which associates with any formula α an element in L and satisfies the following conditions:

(i) $v(\lambda)=\mathbf{0}$.
(ii) $v(\chi)=\mathbf{1/2}$.
(iii) $v(\neg\alpha)=v(\alpha)^{\perp}$.
(iv) $v(\sim\alpha)=v(\alpha)^{\sim}$.
(v) $v(\alpha\wedge b)=v(\alpha)\sqcap v(\beta)$.

Given a BZ lattice with the half element $\mathfrak{L}$, by $\mathrm{Val}(\mathfrak{L})$, we will denote the class of all valuations into $\mathfrak{L}$ and by $R(\mathrm{BZL})$ the class of all algebraic realizations for BZL.

Definition 2.2. (i) A formula α is *true in an algebraic realization* $\mathfrak{U}=\langle\mathfrak{L}, v\rangle$ ($\vDash_{\mathfrak{U}}\alpha$) iff $v(\alpha)=\mathbf{1}$.

(ii) A formula α is a *logical truth* of BZL ($\vDash_{\mathrm{BZL}}\alpha$) iff $\vDash_{\mathfrak{U}}\alpha$ for any $\mathfrak{U}\in R(\mathrm{BZL})$.

(iii) Let T be a set of formulas. We say that a formula α is a *logical consequence* in BZL ($T\vDash\alpha$) iff for any $\langle\mathfrak{L}, v\rangle\in R(\mathrm{BZL})$ and for any $a\in L$: if $a\leq v(\beta)$, $\forall\beta\in T$, then $a\leq v(\alpha)$.

A Kripke semantics for BZL was first proposed in Giuntini (1991). A characteristic feature of this semantics is the use of Kripke frames with two accessibility relations. One can prove Giuntini (1991) that the algebraic and the Kripkean semantics characterize the same logic. BZL can be axiomatized: a soundness and a completeness theorem can be proved with respect to both semantics (Giuntini, 1991).

Characteristic logical properties that fail and hold in BZL are the following:

(a) The distributive principles, the noncontradiction, and the excluded middle principles break down for the fuzzy negation.

(b) As in intuitionistic logic, we have

$$\underset{\mathrm{BZL}}{\vDash}\sim(\alpha\wedge\sim\alpha);\quad \underset{\mathrm{BZL}}{\nvDash}\alpha\vee\sim\alpha;\quad \alpha\underset{\mathrm{BZL}}{\vDash}\sim\sim\alpha;\quad \sim\sim\alpha\underset{\mathrm{BZL}}{\nvDash}\alpha;$$

$$\sim\sim\sim\alpha\underset{\mathrm{BZL}}{\vDash}\sim\alpha;\quad \text{if } \alpha\underset{\mathrm{BZL}}{\vDash}\beta,\quad \text{then } \sim\beta\underset{\mathrm{BZL}}{\vDash}\sim\alpha$$

(c) $\sim\alpha\vDash_{\mathrm{BZL}}\neg\alpha$; $\neg\alpha\nvDash_{\mathrm{BZL}}\sim\alpha$; $\neg\sim\alpha\vDash_{\mathrm{BZL}}\sim\sim\alpha$.

(d) The modal operators give rise to S_5-like behavior:

$$L\alpha\underset{\mathrm{BZL}}{\vDash}\alpha;\quad L(\alpha\wedge\beta)\underset{\mathrm{BZL}}{\vDash}L\alpha\wedge L\beta;\quad L\alpha\wedge L\beta\underset{\mathrm{BZL}}{\vDash}L(\alpha\wedge\beta);$$

$$M(\alpha\wedge\beta)\underset{\mathrm{BZL}}{\vDash}M\alpha\wedge M\beta;\quad L\alpha\underset{\mathrm{BZL}}{\vDash}LL\alpha;\quad M\alpha\underset{\mathrm{BZL}}{\vDash}LM\alpha;$$

$$\text{if }\underset{\mathrm{BZL}}{\vDash}\alpha,\quad \text{then }\underset{\mathrm{BZL}}{\vDash}L\alpha$$

3. A SEMANTICS WITH POSITIVE AND NEGATIVE CERTAINTY DOMAINS

An alternative semantical description for a form of fuzzy intuitionistic logic was first proposed in Cattaneo and Nisticò (1989). The intuitive idea underlying this semantics can be sketched as follows: one supposes that interpreting a language means essentially associating to any sentence two *domains of certainty*: the domain of situations where the sentence certainly holds, and the domain of situations where the sentence certainly does not hold. In contrast to the standard Kripkean semantics, the positive domain of a given sentence does not generally determine the negative domain of the same sentence. As a consequence, propositions are here identified with particular pairs of sets of worlds, rather than with particular sets of worlds as happens in the usual possible-world semantics.

We first present the general construction to obtain BZ lattices of pairs from preclusivity frames (see Definition 3.1) and then we define the notion of *realization with positive and negative certainty domains* for a BZL language.

Definition 3.1. A *preclusivity frame* (or *orthoframe*) is a pair $\mathfrak{F}=\langle W, \#\rangle$, where W is a nonempty set and $\#$ is an irreflexive and symmetric relation on W.

Given an orthoframe $\mathfrak{F}=\langle W, \#\rangle$, define for any $A \subseteq W$:

$$A^{\#} := \{i \in W / i \# j, \forall j \in A\}$$

Let $P(\mathfrak{F}) := \{A \subseteq W / A = A^{\#\#}\}$. We call $P(\mathfrak{F})$ the set of all *simple propositions* of the orthoframe $\mathfrak{F}$.

As is well known, the structure $\mathfrak{P}(\mathfrak{F}) = \langle P(\mathfrak{F}), \subseteq, {}^{\#}, \emptyset, W\rangle$ is a complete ortholattice with maximum (W), minimum ($\emptyset$), where, given any family $\{A_i\}$ of simple propositions

$$\operatorname{Inf}(\{A_i\}) = \bigcap_i A_i \qquad \text{and} \qquad \operatorname{Sup}(\{A_i\}) := \bigsqcup_i A_i = \left(\bigcup_i A_i\right)^{\#\#}$$

Definition 3.2. Let $\mathfrak{F}=\langle W, \#\rangle$ be an orthoframe, X is a *proposition* of $\mathfrak{F}$ iff $X=\langle A, B\rangle$, where $A, B \in P(\mathfrak{F})$ and $A \# B$, i.e., $A \subseteq B^{\#}$.

By $P^{\#}(\mathfrak{F})$, we will denote the class of all propositions of $\mathfrak{F}$.

Definition 3.3. A proposition $\langle A, B\rangle$ is said to be *exact* iff $B = A^{\#}$.

The following operations are defined on the set of all propositions:

(i) The fuzzy complement:
$\langle A, B\rangle^{①} = \langle B, A\rangle$.

(ii) The intuitionistic complement:
$\langle A, B\rangle^{\ominus} = \langle B, B^{\#}\rangle$.

(iii) The propositional conjunction:
$\langle A, B\rangle ⊓ \langle C, D\rangle = \langle A \cap C, B \sqcup D\rangle$.

(iv) The propositional disjunction:
$\langle A, B\rangle ⊔ \langle C, D\rangle = \langle A \sqcup C, B \cap D\rangle$.

(v) The infinitary conjunction:
$⊓ \{\langle A_i, B_i\rangle\} = \langle \bigcap A_i, \bigsqcup B_i\rangle$.

(vi) The infinitary disjunction:
$⊔ \{\langle A_i, B_i\rangle\} = \langle \bigsqcup A_i, \bigcap B_i\rangle$.

(vii) The necessity operator:
$\Box\,(\langle A, B\rangle) = \langle A, A^{\#}\rangle$.

(viii) The possibility operator:
$\Diamond\,(\langle A, B\rangle) = (\Box\,(\langle A, B\rangle^{\oplus}))^{\oplus}$.

(ix) The order relation:
$\langle A, B\rangle \leq (C, D\rangle$ iff $A \subseteq C$ and $D \subseteq B$.

Then, one can easily prove the following theorem.

Theorem 3.1 (Cattaneo and Nisticò, 1989). The structure

$$\mathfrak{P}(\mathfrak{F}) = \langle \mathfrak{P}^{\#}(\mathfrak{F}), \leq, {}^{\oplus}, {}^{\ominus}, \langle \varnothing, W\rangle, \langle W, \varnothing\rangle\rangle$$

is a BZ lattice with maximum ($\langle W, \varnothing\rangle$), minimum ($\langle \varnothing, W\rangle$), and the half element ($\langle \varnothing, \varnothing\rangle$) which satisfies the following conditions:

(i) $(\langle A, B\rangle ⊓ \langle C, D\rangle)^{\ominus} = (A, B\rangle^{\ominus} ⊔ \langle C, D\rangle^{\ominus}$.

(ii) If $\langle A, B\rangle^{\oplus\ominus} \leq \langle C, D\rangle$ and $\langle A, B\rangle \leq \langle C, D\rangle^{\ominus\ominus}$, then

$$\langle A, B\rangle \leq \langle C, D\rangle$$

Let us again assume the BZL language.

Definition 3.4. A *realization with positive and negative certainty domains* (briefly, *orthopair realization* or *Cattaneo realization*) is a system $\mathfrak{M} = \langle W, \#, \Lambda, \sigma\rangle$, where:

(i) $\langle W, \#\rangle$ is an orthoframe, Λ is a subset of the class of all propositions of $\langle W, \#\rangle$, closed under ${}^{\oplus}$, ${}^{\ominus}$, ⊓, and ⊔. Further, Λ must contain the privileged propositions $\mathbf{0}$, $\mathbf{1/2}$ which satisfy the following conditions:

$$\mathbf{0} = \langle \varnothing, W\rangle$$

for any $\langle A, B\rangle \in \Lambda$: $\quad \langle A, B\rangle ⊓ \langle A, B\rangle^{\oplus} \leq \mathbf{1/2}$; $\quad \mathbf{1/2} \leq \mathbf{1/2}^{\oplus}$

(It should be noticed that, in general, $\mathbf{1/2}$ does not coincide with $\langle \varnothing, \varnothing\rangle$.)

(ii) σ is a valuation-function which associates with any formula an element in Λ and satisfies the following conditions:

(a) $\sigma(\lambda) = \mathbf{0}$.

(b) $\sigma(\chi) = \mathbf{1/2}$.

(c) $\sigma(\alpha \wedge \beta) = \sigma(\alpha) ⊓ \sigma(\beta)$.

(d) $\sigma(\neg\alpha)=\sigma(\alpha)^{\oplus}$.
(e) $\sigma(\sim\alpha)=\sigma(\alpha)^{\ominus}$.

The other basic semantical definitions are as in the algebraic semantics for BZL.

As a consequence of Theorem 3.1, one can immediately prove a soundness theorem with respect to orthopair semantics. Further, the propositional lattice $\langle \Lambda, \leq \rangle$ is embedded into the complete lattice $\langle P^{\#}(\mathfrak{F}), \leq \rangle$ of all possible propositions, and the embedding preserves the operations $⊓$, $⊔$, ${}^{\oplus}$, ${}^{\ominus}$.

One might guess that the orthopair semantics characterizes the same logic BZL. However, this conjecture has a negative answer, as the following theorem shows.

Theorem 3.2. For any orthopair realization $\mathfrak{M}=\langle W, \#, \Lambda, \sigma\rangle$,

$$\models_{\mathfrak{M}} \sim\sim(\alpha\wedge\beta)\vee(\sim(\alpha\wedge\beta)\wedge(\sim\alpha\vee\sim\beta))$$

but

$$\not\models_{\mathrm{BZL}} \sim\sim(\alpha\wedge\beta)\vee(\sim(\alpha\wedge\beta)\wedge(\sim\alpha\vee\sim\beta))$$

Proof. Let $\mathfrak{M}=\langle W, \#, \Lambda, \rho\rangle$ be an orthopair realization and let $\sigma(\alpha)=\langle A, B\rangle$ and $\sigma(\beta)=\langle C, D\rangle$. Then $\sigma(\sim\sim(\alpha\wedge\beta))=\langle (B\sqcup D)^{\#}, B\sqcup D\rangle$ and $\sigma(\sim(\alpha\wedge\beta))=\langle B\sqcup D, (B\sqcup D)^{\#}\rangle$. Further,

$$\sigma(\sim\alpha\vee\sim\beta)=\langle B\sqcup D, B^{\#}\cap D^{\#}\rangle$$

Therefore,

$$\begin{aligned}
&\sigma(\sim\sim(\alpha\wedge\beta)\vee(\sim(\alpha\wedge\beta)\wedge(\sim\alpha\vee\sim\beta)))\\
&\quad=\langle (B\sqcup D)^{\#}, B\sqcup D\rangle \,⊔\, (\langle B\sqcup D, B^{\#}\cap D^{\#}\rangle \,⊓\, \langle B\sqcup D, B^{\#}\sqcap D^{\#}\rangle)\\
&\quad=\langle (B\sqcup D)^{\#}, B\sqcup D\rangle \,⊔\, \langle B\sqcup D, (B\sqcup D)^{\#}\rangle\\
&\quad=\langle W, \varnothing\rangle=\mathbf{1}
\end{aligned}$$

Let us now consider the BZ lattice $\mathfrak{P}_{1/2}(\mathfrak{H})$ with the half element of Example 1.5. Let v be a valuation function in $\mathrm{Val}(\mathfrak{P}_{1/2}(\mathfrak{H}))$ such that $v(\alpha)=P$, where $P\notin\{\mathbb{1}, \mathbb{0}, \mathbf{1/2}\cdot\mathbb{1}\}$ and $v(\beta)=\mathbf{1/2}\cdot\mathbb{1}$. Then $v(\sim\sim(\alpha\wedge\beta))=(P\sqcap\mathbf{1/2}\cdot\mathbb{1})^{\sim\sim}=\mathbb{0}^{\sim\sim}=\mathbb{0}$ and $v(\sim(\alpha\wedge\beta))=\mathbb{1}$. Further,

$$v(\sim\alpha\vee\sim\beta)=P^{\sim}\sqcup(\mathbf{1/2}\cdot\mathbb{1})^{\sim}=P^{\perp}\sqcup\mathbb{0}=P^{\perp}$$

Thus,

$$v(\sim\sim(\alpha\wedge\beta)\vee(\sim(\alpha\wedge\beta)\wedge(\sim\alpha\vee\sim\beta)))=\mathbb{0}\sqcup(\mathbb{1}\sqcap P^{\perp})=P^{\perp}\neq\mathbb{1} \quad \blacksquare$$

As a consequence, the orthopair semantics characterizes a logic which is stronger than BZL. We will call this logic BZL*. This logic can be axiomatized: a soundness and a completeness theorem can be proved with respect to the orthopair semantics (Cattaneo *et al.*, 1990).

4. ALGEBRAIC SEMANTICS FOR BZL*

In this section, we show that BZL* can be characterized also by means of algebraic semantics and we prove that there are some conditions which make a BZ lattice embeddable into the BZ lattice of all propositions of an orthoframe.

Definition 4.1. A BZ* lattice is a BZ lattice $\mathfrak{L}=\langle L, \leq, {}^{\perp}, {}^{\sim}, \mathbf{0}, \mathbf{1}\rangle$ with the half element such that the following conditions are satisfied:

(i) $\forall a, b \in L$: $(a \sqcap b)^{\sim} = a^{\sim} \sqcup b^{\sim}$.

(ii) $\forall a, b \in L$: if $a^{\perp\sim} \leq b$ and $a \leq b^{\sim\sim}$, then $a \leq b$.

Lemma 4.1. The class of all BZ* lattices is equational.

Proof. We will show that (ii) is equivalent to the following equality:

$$a \sqcap b^{\sim\sim} \leq a^{\perp\sim} \sqcup b \tag{$*$}$$

Suppose that (ii) holds. We want to show that $(a \sqcap b^{\sim\sim})^{\perp\sim} \leq a^{\perp\sim} \sqcup b$ and $(a \sqcap b^{\sim\sim}) \leq (a^{\perp\sim} \sqcup b)^{\sim\sim}$. Then, by (ii) we can conclude that $(*)$ holds. $(a \sqcap b^{\sim\sim})^{\perp\sim} = a^{\perp\sim} \sqcap b^{\sim\sim} \leq a^{\perp\sim} \sqcup b$. On the other hand,

$$(a^{\perp\sim} \sqcup b)^{\sim\sim} = a^{\perp\sim} \sqcup b^{\sim\sim} \geq b^{\sim\sim} \sqcap a$$

Conversely, suppose $a^{\perp\sim} \leq b$ and $a \leq b^{\sim\sim}$. Then, by $(*)$,

$$a = a \sqcap b^{\sim\sim} \leq a^{\perp\sim} \sqcup b = b \quad \blacksquare$$

We list, without proof, some properties of a BZ* lattice $\mathfrak{L}$:

$(*1)$ $\forall a \in L$: $a \leq a^{\perp}$ iff $a^{\perp\sim} = \mathbf{0}$.

$(*2)$ $a \leq a^{\perp}$ iff $a \leq \mathbf{1/2}$.

$(*3)$ $a \sqcap b^{\perp\sim\sim} \leq a^{\perp\sim} \sqcup b^{\perp}$.

Theorem 4.1. Conditions (i) and (ii) of Definition 4.1 are independent.

Proof. First, we prove that there exists a BZ lattice with the half element in which condition (i) holds and condition (ii) fails. Let $\mathfrak{L}_{[0,1]}$ be the BZ lattice of Example 1.4. Two cases are possible: (a) $(a \sqcap b)^{\sim} = \mathbf{1}$; (b) $(a \sqcap b)^{\sim} = \mathbf{0}$.

(a) If $(a \sqcap b)^{\sim} = \mathbf{1}$, then $\mathbf{0} = a \sqcap b = \text{Min}(\{a, b\})$. Thus, either $a = \mathbf{0}$ or $b = \mathbf{0}$. In both cases, $a^{\sim} \sqcup b^{\sim} = \mathbf{1}$.

(b) If $(a \sqcap b)^{\sim} = \mathbf{0}$, then $a \sqcap b \neq \mathbf{0}$. Thus, both $a \neq \mathbf{0}$ and $b \neq \mathbf{0}$. Therefore, $a^{\sim} = b^{\sim} = \mathbf{0}$. Hence, $a^{\sim} \sqcup b^{\sim} = \mathbf{0}$.

The equational condition equivalent to (ii) fails in $\mathfrak{L}_{[0,1]}$. Let $a=2/3$ and $b=1/3$. Then

$$a\sqcap b^{\sim\sim}=2/3\sqcap \mathbf{1}=2/3\not\leq 1/3=\mathbf{0}\sqcup 1/3=a^{\perp\sim}\sqcup b$$

Now, we can prove that there exists a BZ lattice with the half element where condition (ii) holds and condition (i) fails. Let us consider the BZ lattice $\mathfrak{P}_{1/2}(\mathfrak{H})$ of Example 1.5. We want to show that $\forall E, F\in P_{1/2}(\mathfrak{H})$: $E\sqcap F^{\sim\sim}\leq E^{\perp\sim}\sqcup F$. If E, F are two projectors, then the proof is trivial. Thus, we can suppose that either $E=\mathbf{1/2}\cdot \mathbb{1}$ or $F=\mathbf{1/2}\cdot \mathbb{1}$. Suppose that $E=\mathbf{1/2}\cdot \mathbb{1}$. If $F\in\{\mathbb{1}, \mathbb{0}, 1/2\cdot \mathbb{1}\}$, then the proof is trivial. Therefore, we can suppose that $F\notin\{\mathbb{1}, \mathbb{0}, \mathbf{1/2}\cdot \mathbb{1}\}$. Then $E\sqcap F^{\sim\sim}=E\sqcap F=\mathbf{0}$ and we are done. The case in which $F=\mathbf{1/2}\cdot \mathbb{1}$ is similar.

We now prove that $(E\sqcap F)^{\sim}\not\leq E^{\sim}\sqcup F^{\sim}$. Let $E\notin\{\mathbb{1}, \mathbb{0}, \mathbf{1/2}\cdot \mathbb{1}\}$ and let $F=\mathbf{1/2}\cdot \mathbb{1}$. Then $(E\sqcap F)^{\sim}=\mathbf{1}$ and $E^{\sim}=E^{\perp}\neq\mathbf{1}$. ■

Definition 4.2. Let $\mathfrak{L}$ be a BZ lattice. An L-*filter* of $\mathfrak{L}$ is a filter F of $\mathfrak{L}$ which satisfies the following condition:

$$\forall a\in L:\quad \text{if } a\in F,\text{ then } a^{\perp\sim}\in F$$

One can easily prove the following result.

Lemma 4.2. Let $\mathfrak{L}$ be a BZ lattice. The following properties hold:

(i) $[a^{\perp\sim}\rangle$ and $[a^{\sim\sim}\rangle$ (the principal filters generated by $a^{\perp\sim}$ and $a^{\sim\sim}$) are L-filters.

(ii) An L-filter F is proper iff $\forall a\in L$: $a\sqcap a^{\perp}\notin F$.

Theorem 4.2. Let $\mathfrak{L}=\langle L, \leq, {}^{\perp}, {}^{\sim}, \mathbf{0}, \mathbf{1}\rangle$ be a BZ lattice with the half element. Then the following conditions are equivalent.

(i) $\mathfrak{L}$ is embeddable into the BZ lattice of all propositions of a preclusivity frame.

(ii) $\mathfrak{L}$ is a BZ* lattice.

Proof. (i) implies (ii). It suffices to prove that the BZ lattice $\mathfrak{P}^{\#}(\mathfrak{F})$ of all possible propositions of an orthoframe $\mathfrak{F}=\langle W, {}^{\#}\rangle$ satisfies conditions (i) and (ii) of Definition 4.1. Conditions (i) and (ii) follow from Theorem 3.1.

(ii) implies (i). Let W be the class of all proper L-filters. Given $F, G\in W$, define $F\# G$ iff $\exists a\in L$ such that $a\in F$ and $a^{\perp}\in G$. Clearly, $\#$ is symmetric and, by Lemma 4.2(ii), irreflexive. Therefore, the pair $\mathfrak{F}=\langle W, \#\rangle$ is an orthoframe. Let $\mathfrak{P}(\mathfrak{F})$ be the ortholattice of all simple propositions of $\mathfrak{F}$ and let $\mathfrak{P}^{\#}(\mathfrak{F})$ be the BZ lattice of all propositions of $\mathfrak{F}$. Let h be the map from $\mathfrak{L}$ into 2^{W} defined as follows: $h(a)=\{F\in W/a\in F\}$. We want to show that, for any $a\in L$, $h(a)$ is a simple proposition. If $a=\mathbf{0}$, then $h(a)=\varnothing=\varnothing^{\#\#}$. If $a=1$, then $h(a)=W=W^{\#\#}$. Therefore, we can suppose $a\neq\mathbf{0}$ and $a\neq\mathbf{1}$.

Suppose $F \notin h(a)$. We want to prove there exists a $G \in W$ such that not $F \# G$ and $G \in h(a)^{\#}$. Let us consider the filter $G := [a^{\perp\sim\sim}\rangle$. By Lemma 4.2(i), G is an L-filter. G is proper since otherwise $\mathbf{1} = \mathbf{0}^{\sim} = a^{\perp\sim} \leq a$, impossible since $a \neq \mathbf{1}$. Thus, $G \in W$. Now, we can prove that $G \in h(a)^{\#}$. Suppose $H \in h(a)$. Then $a \in H$ and therefore $a^{\perp\sim} \in H$ since H is an L-filter. But $a^{\perp\sim\perp} = a^{\perp\sim\sim} \in G$. Thus, $G \# H$. It remains to prove that not $F \# G$. Suppose, by contradiction, that $F \# G$. Then there exists an element b such that $a^{\perp\sim\sim} \leq b$ and $b^{\perp} \in F$. Then, $a^{\perp\sim} \in F$; hence, $a \in F$, which contradicts the hypothesis $F \notin h(a)$. Therefore, $h(a) \in P(\mathfrak{F})$.

Let us define the map $k: L \to P(\mathfrak{F}) \times P(\mathfrak{F})$ as follows: $k(a) = \langle h(a), h(a^{\perp}) \rangle$. An easy computation shows that $h(a) \subseteq h(a^{\perp})^{\#}$. Therefore, k maps $\mathfrak{L}$ into $\mathfrak{P}^{\#}(\mathfrak{F})$, the BZ lattice of all propositions of $\mathfrak{F}$. We want to show that k is an embedding of $\mathfrak{L}$ into $\mathfrak{P}^{\#}(\mathfrak{F})$.

(a) $k(a^{\perp}) = k(a)^{\textcircled{\scriptsize $\perp$}}$. We have

$$k(a^{\perp}) = \langle h(a^{\perp}), h(a) \rangle = k(a)^{\textcircled{\scriptsize $\perp$}}$$

(b) $k(a^{\sim}) = k(a)^{\textcircled{\scriptsize $\sim$}}$. By definition, $k(a^{\sim}) = \langle h(a^{\sim}), h(a^{\sim\sim}) \rangle$ and $k(a)^{\textcircled{\scriptsize $\sim$}} = \langle h(a^{\perp}), h(a^{\perp})^{\#} \rangle$. First, we prove that $h(a^{\sim}) = h(a^{\perp})$. Now, $h(a^{\sim}) \subseteq h(a^{\perp})$ since $a^{\sim} \leq a^{\perp}$. Suppose $F \in h(a^{\perp})$. Then $a^{\sim} = a^{\perp\perp\sim} \in F$, since F is an L-filter. $h(a^{\sim\sim}) \subseteq h(a^{\perp})^{\#}$. Suppose $F \in h(a^{\sim\sim})$ and $G \in h(a^{\perp})$. We want to prove that $F \# G$. By hypothesis, $a^{\sim\sim} \in F$ and $a^{\perp} \in G$. Then $a^{\sim\sim\perp} = a^{\sim} = a^{\perp\perp\sim} \in F$ since F is an L-filter. $h(a^{\perp})^{\#} \subseteq h(a^{\sim\sim})$. Suppose $F \in h(a^{\perp})^{\#}$. Then $F \# G$ for any L-filter G such that $G \in h(a^{\perp})$. Let $H := [a^{\sim}\rangle$. If $a^{\sim} = \mathbf{0}$, then $a^{\sim\sim} = \mathbf{1}$ so that $F \in h(a^{\sim\sim})$. Therefore, we can suppose $a^{\sim} \neq \mathbf{0}$. By Lemma 4.2, H is a proper L-filter. Now, $a^{\perp} \in H$, since $a^{\sim} \leq a^{\perp}$. Then, by hypothesis, $F \# H$. Thus, $\exists b \in L$ such that $b \in F$ and $b^{\perp} \in H$, i.e., $a^{\sim} \leq b^{\perp}$. Therefore, $b \leq a^{\sim\sim}$; hence $F \in h(a^{\sim\sim})$.

(c) $k(a \sqcap b) = k(a) \textcircled{\scriptsize $\sqcap$} k(b)$. By definition,

$$k(a \sqcap b) = \langle h(a \sqcap b), h(a^{\perp} \sqcup b^{\perp}) \rangle$$

and

$$k(a) \textcircled{\scriptsize $\sqcap$} k(b) = \langle h(a) \sqcap h(b), h(a^{\perp}) \sqcup h(b^{\perp}) \rangle$$

It is easy to see that $h(a \sqcap b) = h(a) \sqcap h(b)$. It remains to prove that $h(a^{\perp} \sqcup b^{\perp}) = h(a^{\perp}) \sqcup h(b^{\perp})$. We will prove that, in general, the following equality holds:

$$h(a \sqcup b) = h(a) \sqcup h(b) \tag{$*$}$$

To prove $(*)$ it suffices to show that

$$\forall F \in W (F \in h(a \sqcup b) \text{ iff } \forall G \in W (\text{if not } F \# G,$$
$$\text{then there exists an } H \in W \text{ such that}$$
$$\text{not } G \# H \text{ and } G \in h(a) \text{ or } G \in h(b)))$$

Suppose $F \in h(a \sqcup b)$. Then $(a \sqcup b)^{\perp\sim} \in F$ since F is an L-filter. Now, $\mathfrak{L}$ is a BZ* lattice and therefore $a^{\perp\sim} \sqcup b^{\perp\sim} = (a \sqcup b)^{\perp\sim} \in F$. Suppose, not $F \# G$. We want to show that either $a^{\perp\sim\sim} \notin G$ or $b^{\perp\sim\sim} \notin G$. Suppose, by contradiction, that $a^{\perp\sim\sim} \in G$ and $b^{\perp\sim\sim} \in G$. Then, $(a^{\perp\sim} \sqcup b^{\perp\sim})^{\perp} = a^{\perp\sim\sim} \sqcap b^{\perp\sim\sim} \in G$. Then $F \# G$, contradiction. Suppose $a^{\perp\sim\sim} \notin G$. Then $a^{\perp\sim} \neq \mathbf{0}$. Thus, by Lemma 4.2, $H := [a^{\perp\sim}\rangle$ is a proper L-filter. Moreover, $H \in h(a)$. It remains to prove that not $H \# G$. Suppose, by contradiction, that $H \# G$. Then $\exists c \in L$ such that $a^{\perp\sim} \leq c$ and $c^{\perp} \in G$. Then $a^{\perp\sim\sim} \in G$, which contradicts the hypothesis. Suppose $F \notin h(a \sqcup b)$. We want to show that there exists a proper L-filter G such that not $F \# G$ and $\forall H \in W$: if not $H \# G$, then $H \notin h(a)$ and $H \notin h(b)$.

By hypothesis, $a \sqcup b \notin F$. We want to show that $(a^{\perp} \sqcap b^{\perp})^{\sim\sim} \neq \mathbf{0}$. Suppose, by contradiction, that $(a^{\perp} \sqcap b^{\perp})^{\sim\sim} = \mathbf{0}$. Then $(a \sqcup b)^{\perp\sim} = \mathbf{1} \in F$. Then $a \sqcup b \in F$, impossible. Thus, by Lemma 4.2(i), $G := [(a^{\perp} \sqcap b^{\perp})^{\sim\sim}\rangle$ is a proper L-filter. We want to show that not $F \# G$. Suppose, on the contrary, that $\exists b \in L$ such that $b \in F$ and $(a^{\perp} \sqcap b^{\perp})^{\sim\sim} \leq b^{\perp}$. Then, since $\mathfrak{L}$ is a BZ* lattice, $b \leq (a^{\perp} \sqcap b^{\perp})^{\sim} = a^{\perp\sim} \sqcup b^{\perp\sim}$. Thus, $a \sqcup b \in F$, which contradicts the hypothesis. Suppose, now, not $H \# G$. We want to prove that $H \notin h(a)$ and $H \notin h(b)$. Suppose, by contradiction, $a \in H$ or $b \in H$. Suppose $a \in H$ (the case $b \in H$ is similar). Then $a^{\perp\sim} \in H$, since H is an L-filter. Since $\mathfrak{L}$ is a BZ* lattice, $a^{\perp\sim\sim} \sqcap b^{\perp\sim\sim} = (a^{\perp} \sqcap b^{\perp})^{\sim\sim} \in G$. Then $a^{\perp\sim\perp} \in G$, so that $G \# H$, contradiction. Thus, we have proved that $k(a \sqcup b) = k(a) \circledcirc k(b)$.

By (a)–(c), we can conclude that k is as homomorphism of $\mathfrak{L}$ into $\mathfrak{P}^{\#}(\mathfrak{F})$.

It remains to show that the map k is injective.

First, we will prove that for any $a, b \in L$: if $h(a) \subseteq h(b)$, then $a^{\perp\sim} \leq b$. Suppose $h(a) \subseteq h(b)$. We can suppose $a^{\perp\sim} \neq \mathbf{0}$. Then, by Lemma 4.2, $F := [a^{\perp\sim}\rangle$ is a proper L-filter such that $F \in h(a)$. By hypothesis, $F \in h(b)$, i.e., $a^{\perp\sim} \leq b$.

Suppose, now, that $k(a) = k(b)$. Then $h(a) = h(b)$ and $h(a^{\perp}) = h(b^{\perp})$. As previously shown, $a^{\perp\sim} \leq b$, $a^{\sim} = a^{\perp\perp\sim} \leq b^{\perp}$ and $b^{\perp\sim} \leq a$, $b^{\sim} \leq a^{\perp}$. Thus, $a^{\perp\sim} \leq b$, $a \leq b^{\sim\sim}$, and $b^{\perp\sim} \leq a$, $b \leq a^{\sim\sim}$. Since $\mathfrak{L}$ is a BZ* lattice, we can conclude, by condition (ii) of Definition 4.1, $a \leq b$ and $b \leq a$. ■

The definition of algebraic realization for BZL* and the basic semantical definitions are as in the algebraic semantics for BZL.

We can now prove that the algebraic semantics and the Cattaneo semantics characterize the same logic.

By

$$T \overset{\text{A}}{\models} \alpha$$

we mean that α is a logical consequence of T according to the algebraic semantics.

By

$$T \overset{\mathrm{C}}{\models} \alpha$$

we mean that α is a logical consequence of T according to the orthopair semantics.

Theorem 4.3

$$T \overset{\mathrm{A}}{\models} \alpha \quad \text{iff} \quad T \overset{\mathrm{C}}{\models} \alpha$$

The proof of Theorem 4.3 is a direct consequence of the following lemmas.

Lemma 4.3. For any algebraic realization $\mathfrak{U} = \langle \mathfrak{L}, v \rangle$ for BZL*, there exists an orthopair realization $\mathfrak{M}^{\mathfrak{U}} = \langle W, \#, \Lambda, \sigma \rangle$ such that

$$T \models_{\mathfrak{U}} \alpha \quad \text{iff} \quad T \models_{\mathfrak{M}^{\mathfrak{U}}} \alpha$$

Lemma 4.4. For any orthopair realization $\mathfrak{M} = \langle W, \#, \Lambda, \sigma \rangle$, there exists an algebraic realization $\mathfrak{U}^{\mathfrak{M}} = \langle \mathfrak{L}, v \rangle$ for BZL* such that

$$T \models_{\mathfrak{M}} \alpha \quad \text{iff} \quad T \models_{\mathfrak{M}} \alpha$$

Proof of Lemma 4.3. Let $\mathfrak{U} = \langle \mathfrak{L}, v \rangle$ be an algebraic realization.

Define W as the class of all proper L-filters of $\mathfrak{L}$ and $\#$ as in Theorem 4.2.

By Theorem 4.2, we know that the BZ* lattice $\mathfrak{L}$ is embeddable in the BZ* lattice $\mathfrak{P}^{\#}(\mathfrak{F})$ of all possible propositions of the preclusivity frame $\mathfrak{F} = \langle W, \# \rangle$. Let k be such an embedding. Define $\Lambda = \{k(a)/a \in L\}$ and $\sigma(\alpha) = k(v(\alpha))$. Then it is easy to check that the system $\mathfrak{M} = \langle W, \#, \Lambda, \sigma \rangle$ is a "good" orthopair realization.

The proof of the fact that

$$T \models_{\mathfrak{U}} \alpha \quad \text{iff} \quad T \models_{\mathfrak{M}^{\mathfrak{U}}} \alpha$$

is a consequence of the fact that k is an embedding. ■

Proof of Lemma 4.4. Let $\mathfrak{M} = \langle W, \#, \Lambda, \sigma \rangle$ be an orthopair realization.

By Theorem 2.2, we know that Λ is a BZ* lattice. Define $v(\alpha) = \sigma(\alpha)$.

Then $\langle \Lambda, v \rangle$ is an algebraic realization for BZL* such that

$$T \models_{\mathfrak{U}} \alpha \quad \text{iff} \quad T \models_{\mathfrak{M}^{\mathfrak{U}}} \alpha$$

Some interesting questions which can be investigated in connection with this logic are the following:

(1) A Kripkean semantic characterization of BZL*.

(2) Possible models of BZL* based on subsets of effects of a Hilbert space.

(3) Finite model property and decidability for BZL*.

(4) Orthomodular extensions of BZL*. ■

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